NON-UNIFORM SPECIFICATION AND LARGE DEVIATIONS FOR WEAK GIBBS MEASURES

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Abstract. We establish bounds for the measure of deviation sets associated to continuous observables with respect to not necessarily invariant weak Gibbs measures. Under some mild assumptions, we obtain upper and lower bounds for the measure of deviation sets of some non-uniformly expanding maps, including quadratic maps and robust multidimensional non-uniformly expanding local diffeomorphisms. For that purpose, a measure theoretical weak form of specification is introduced and proved to hold for the robust classes of multidimensional nonuniformly expanding local diffeomorphisms considered in [VV10] and Viana maps [Via97].

1. Introduction

The theory of Large Deviations concerns the study of the rates of convergence at which time averages of a given sequence of random variables converge to the limit distribution. An application of these ideas into the realm of Dynamical Systems is useful to estimate the velocity at which typical points of ergodic invariant measures converge to the corresponding space averages. More generally, given a continuous transformation f on a compact metric space M and a reference measure ν, one would like to provide sharp estimates for the ν-measure the deviation sets

$$\left\{ x \in M : \frac{1}{n} \sum_{j=0}^{n-1} g(f^j(x)) > c \right\}$$

for all continuous functions $g : M \to \mathbb{R}$ and real numbers c. To this purpose, a priori estimates on the measure of the dynamical balls

$$B(x, n, \varepsilon) = \big\{ y \in M : d(f^j(y), f^j(x)) \le \varepsilon, \ \forall \, 0 \le j \le n \big\}$$

for $x \in M$, $\varepsilon > 0$ and $n \ge 1$ are useful and somewhat necessary since points that belong to the same dynamical ball have nearby Birkhoff averages with respect to continuous functions.

Some large deviations ideas and techniques are particularly useful to the study of the thermodynamical formalism of transformations with some hyperbolicity. Recall that the variational principle for the pressure asserts that for every continuous potential ϕ

$$P_{\text{top}}(f, \phi) = \sup \left\{ h_\eta(f) + \int \phi \, d\eta \right\},$$

where the supremum is taken over all invariant probability measures η. A measure μ that attains the supremum in the variational principle is called an equilibrium state for f with respect to the potential ϕ. A large deviations theory was developed for

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uniformly hyperbolic systems restricted to a basic piece of the non-wandering set and Hölder continuous potentials in both discrete and time-continuous settings. Indeed, such hyperbolic transformations admit a unique equilibrium state with respect to any Hölder continuous potential (see [Sin72, Bow75, Rue76]), Young, Kifer and Newhouse [You90, Kif90, KN91]) established, in the mid nineties, large deviation principles for this important open class of dynamical systems: the rate of decay is given explicitly in terms of the distance of all invariant measures η with "bad" space averages to the equilibrium state μ. Focusing on the discrete time case, the sharp lower and upper bounds obtained in [You90] for the measure of deviation sets yield as a consequence that for any ergodic equilibrium state μ and every continuous observable g, the measure of the set of points whose time average $\frac{1}{n}\sum_{j=0}^{n-1} g(f^j(x))$ is far from the space average $\int g\, d\mu$ decreases exponentially fast. One key ingredient to obtain the large deviations principle is that, when restricted to a basic piece of the non-wandering set, every uniformly hyperbolic dynamical system is semi-conjugated to a subshift of finite type that satisfies a very "strong mixing" condition known as specification. This notion, introduced by Bowen [Bow71], means roughly that any finite sequence of pieces of orbit can be well approximated by periodic ones. We also point out that sharp lower bound estimate for the measure of the deviation sets in terms of the free energy may fail to exist even in the uniformly hyperbolic case due to the non-uniqueness of equilibrium states (see [Kif90]).

Since the nineties many efforts have been made in the attempt to extend the theory of large deviations to the scope of non-uniformly hyperbolic dynamics and some important results in that direction have been obtained recently. Araújo and Pacífico [AP06] established large deviation upper bounds for the deviation sets of physical measures for non-uniformly expanding maps (in the sense of [ABV00]). More recently, Melbourne and Nicol [MN08, Mel09] studied systems that admit some inducing Markov structure, and proved that the measure of points with atypical time averages for a Hölder continuous potential has the same decay rate as the inducing time itself. In particular, less than exponential rate of convergence to equilibrium is studied. Independently, in the case of exponential tail, Rey-Bellet and Young [RBY08] obtained similar and sharper results. The construction of (countable) expanding Markov maps in [Pin09] provides many examples where the previous results apply. Large deviations principles were also obtained by Yuri (see [Yur05]) in the context of shifts with countably many symbols and by Comman and Rivera (see [CRL98]) for non-uniformly expanding rational maps. However, the previous results give estimates for the measure of deviation sets only with respect to the invariant probability measure.

Our purpose here is to obtain large deviations estimates with respect to not necessarily invariant weak Gibbs, which appear naturally in the study of the thermodynamical formalism in the non-uniformly hyperbolic context. These reference measures are relevant from the physical point of view since they are observable. Furthermore, in many relevant situations, equilibrium states arise as invariant measures absolutely continuous with respect to some reference measure endowed with the weak Gibbs property. Inspired by the pioneering work of Young [You90] we first present an abstract criterium and deviation estimates with respect to an arbitrary reference weak Gibbs measure. Then, under some mild assumptions, large deviations estimates are derived for non-uniformly expanding maps with respect to weak Gibbs measures. Roughly, one proves that the set of points whose time

averages remain far from the space average with respect to the equilibrium decrease exponentially fast with a decay rate related to the existence of invariant expanding measures with frequent hyperbolicity (we refer the reader to Theorem 2.2 for the precise statement). Some of the difficulties arising from the lack of uniform hyperbolicity were overcome using the mild control on the measure of dynamical balls given by the weak Gibbs measure, which cannot be taken uniform in general. In consequence, in the upper bound estimates the loss of uniformity appears expressed in terms of the decay of the first hyperbolic time map. For the lower bound estimates the picture changes considerably, since these are usually associated to some approximation of invariant measures by a finite convex combination of ergodic ones together with the specification property. Despite the fact that the later topological property holds for topologically mixing interval maps and dynamical systems with arbitrary small finite Markov partition, conceptually one cannot expect this to hold with great generality in the absence of uniform hyperbolicity. So, we deduce lower bounds for the measure of the deviation set under a mild specification property which one expects to hold generally in the non-uniformly hyperbolic context. One technical difficulty to overcome (see the claim in the proof of Proposition 5.1 below) is that no a priori estimates for the measure of dynamical balls hold for specified orbits even if the dynamical system satisfies the specification property.

Finnaly, we note that the measure theoretical non-uniform specification property introduced here is proved to hold for a large class of robust nonuniformly expanding maps as in [VV10] and the multidimensional nonuniformly hyperbolic attractors with critical region considered in [Via97], that may not satisfy the strong specification property. This class of transformations seem to constitute first multidimensional examples presenting a weak form of specification in a nonuniformly hyperbolic context.

This paper is organized as follows. Our main results are stated along Section 2. In Section 3 we recall some necessary definitions and prove some preliminary lemmas. The proofs of our main results are given in Sections 4 and 5. Finally, in Section 6 we present some examples and further questions.

2. Statement of results

2.1. **Abstract Theorem.** Let $f : M \to M$ be a continuous transformation on a compact metric space M and let ν be some (not necessarily invariant) probability measure. In this section we state an abstract result on the deviation of Birkhoff averages given by continuous observables.

Given an observable $\phi : M \to \mathbb{R}$, we denote by $S_n\phi(x) = \sum_{j=0}^{n-1} \phi \circ f^j$ the nth Birkhoff sum of ϕ. Given a full ν-measure set $\Lambda \subset M$, denote by $\mathcal{F}(\Lambda)$ the set of continuous functions $\psi \in C(M, \mathbb{R})$ so that, there exists $\delta_0 > 0$ and for every $x \in \Lambda$ and $0 < \varepsilon < \delta_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \frac{1}{n} \log K_n(x, \varepsilon) = 0$ and

$$K_n(x, \varepsilon)^{-1}\, e^{-S_n\psi(x)} \leq \nu(B(y, n, \varepsilon)) \leq K_n(x, \varepsilon)\, e^{-S_n\psi(x)} \tag{2.1}$$

for every $n \geq 1$ and every $y \in M$ satisfying $B(y, n, \varepsilon) \subset B(x, n, \delta_0)$. In most cases the set Λ cannot be taken compact and the previous convergence fails to be uniform in x and ε. We define also $\delta(\varepsilon, \beta)$ as the exponential decay rate at which the ν-measure of the points whose constants K_n grow at most β-exponentially, that is,

if

$$\Delta_n = \Big\{ x \in \Lambda : K_n(x,\varepsilon) < e^{\beta n} \Big\}, \tag{2.2}$$

then set $\delta(\varepsilon,\beta) = \limsup_{n\to\infty} \frac{1}{n} \log \nu(\Delta_n^c)$. In a context of nonuniform hyperbolicity the quantity $\delta(\varepsilon,\beta)$ appears as the exponential decay of the instants of hyperbolicity, does not depend on ε and it is negative for interesting class of examples that appear in Section 6. Finally, the *relative entropy* of an f-invariant probability measure η is defined as $h_\nu(f,\eta) = \eta\text{-esssup}\, h_\nu(f,\cdot)$, where

$$h_\nu(f,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \nu(B(x,n,\varepsilon)), \quad \text{for all } x \in M.$$

We will also need the following:

Definition 2.1. *We say that a map f satisfies the* specification property *if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any sequence of positive integers $n_1, \dots, n_k$ and $p_1, \dots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that*

$$d\Big(f^j(x), f^j(x_1)\Big) \leq \varepsilon, \quad \forall\, 0 \leq j \leq n_1$$

and

$$d\Big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\Big) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Note that this notion of specification is purely topological and is slightly weaker than the one introduced by Bowen [Bow71], that requires that any finite sequence of pieces of orbit is well approximated by periodic orbits. In fact, this condition is known to imply that the system is topologically mixing [Bow71]. It might seem that specification is quite rare among most dynamical systems. However, Blokh [Blo83] proved in a surprising way that the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping. This is no longer true if the one-dimensional map fails to be continuous (see e.g. [Buz97]). We refer the reader to [Wal82] for more details on the specification property. Our first result is as follows.

Theorem 2.1. *Assume that $h_{\text{top}}(f) < \infty$, let ν be a probability measure and let $\Lambda \subset M$ be such that $\nu(\Lambda) = 1$. Given $g \in C(M,\mathbb{R})$ and $c \in \mathbb{R}$, if $\psi \in \mathcal{F}(\Lambda)$ then for every small $\varepsilon, \beta > 0$ it holds*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\Big[x \in M : \frac{1}{n} S_n g(x) \geq c\Big] \leq \max\Big\{\delta(\varepsilon,\beta)\ , \sup\big\{h_\eta(f) - \int \psi\, d\eta\big\} + \beta\Big\}$$

where the supremum is over all invariant probability measures η such that $\int g\, d\eta \geq c$. Moreover, it holds that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c\right) \geq \sup\Big\{h_\eta(f) - h_\nu(f,\eta)\Big\},$$

where the supremum is taken over all ergodic *measures η satisfying $\int g\, d\eta > c$. Furthermore if $\psi \in \mathcal{F}(\Lambda)$ and f satisfies the specification property then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c\right) \geq \sup\left\{h_\eta(f) - \int \psi\, d\eta\right\},$$

where the supremum is taken over all invariant probability measures η such that $\eta(\Lambda) = 1$ and $\int g\, d\eta > c$.

This theorem generalizes Theorem 1 in [You90], where $\Lambda = M$ was assumed to be compact and some uniform control on the measure of partition elements was required.

2.2. **Deviation bounds for non-uniformly expanding maps.**

2.2.1. *Context.* Let M be a compact Riemaniann manifold and let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside of a compact critical or singular region $\mathcal{C}$. Assume:

(H) *f behaves like a power of the distance to the critical or singular set $\mathcal{C}$*: there exist $B > 1$ and $\beta \in (0,1)$ such that for every $x, y \in M \setminus \mathcal{C}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{C})/2$ and every $v \in T_xM$:

  (a) $\frac{1}{B}\operatorname{dist}(x,\mathcal{C})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\operatorname{dist}(x,\mathcal{C})^{-\beta}$;

  (b) $\big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\,\big| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{C})^{\beta}}$;

  (c) $\big|\log|\det Df(x)^{-1}| - \log|\det Df(y)^{-1}|\,\big| \le B\frac{\operatorname{dist}(x,y)}{\operatorname{dist}(x,\mathcal{C})^{\beta}}$.

This condition was proposed in [ABV00] as a multidimensional counterpart of the non-flat critical points in one-dimensional dynamics. We also assume the following condition on f:

(C) There exists $L > 0$ and $\gamma \in (0,1)$ such that for any small $\varepsilon > 0$ every connected component in the preimage of a set of diameter ε is contained in a ball of radius $L\varepsilon^{\gamma}$.

This condition is clearly satisfied if f is a local diffeomorphism and, since f behaves like a power of the distance to $\mathcal{C}$, it is most likely to hold e.g. if $\mathcal{C}$ has empty interior. Such condition is satisfied by the class transformations with singularities (quadratic and Viana maps) considered in Section 6. Let $\phi : M\backslash\mathcal{C} \to \mathbb{R}$ be an Hölder continuous potential and assume:

(P1) There exists a probability measure ν that is positive on open sets, it is non-singular with respect to f with Hölder continuous Jacobian $J_\nu f = \lambda e^{-\phi}$, for some $\lambda > 0$. We will refer to ν as a *conformal measure* associated to ϕ;

(P2) (f,ν) has *non-uniform expansion*: there exists $\sigma > 1$ such that for ν-a.e. x

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df(f^j(x)^{-1}\| \le -2\log\sigma < 0$$

and

$$(\forall\varepsilon > 0)\,(\exists\delta > 0)\ \limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} -\log d_\delta(f^j(x),\mathcal{C}) < \varepsilon.$$

These assumptions are quite natural in a context of non-uniform hyperbolicity and are verified by a large class of maps and potentials. For instance, if f is a non-uniformly expanding map (in the sense of [ABV00]) and $\phi = -\log|\det Df|$ then the Lebesgue measure is a conformal measure that satisfies (P1) and (P2). Usually conformal measures appear as eigenmeasures associated to the dual $\mathcal{L}_\phi^*$ of the Ruelle-Perron-Frobenius operator

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)}\,g(y),$$

acting on the space of probability measures $\mathcal{M}$. Moreover, hypothesis (P1) and (P2) together with the fact that the potential ϕ is Hölder continuous yield that ν is a *weak Gibbs measure*: there are $P \in \mathbb{R}$ and $\delta > 0$ so that for any $0 < \varepsilon \leq \delta$ and almost every x there is a sequence of positive numbers $(K_n)_{n\geq 1}$ (depending also on ϕ) satisfying $\lim_{n\to\infty} \frac{1}{n} \log K_n(x, \varepsilon) = 0$ and

$$K_n(x,\varepsilon)^{-1} \leq \frac{\nu(B(x,n,\varepsilon))}{e^{-Pn+S_n\phi(y)}} \leq K_n(x,\varepsilon) \tag{2.3}$$

for every $y \in B(x, n, \varepsilon)$ (see e.g. [VV10]). Compare to Lemma 3.3 and Corollary 3.2 below. We say that n is a (σ, δ)-*hyperbolic time* for $x \in M$ (or hyperbolic time for short) if there is a small positive constant $b > 0$ such that

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x)^{-1}\| \leq \sigma^{-k} \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{C}) > \sigma^{-bk}$$

for every $1 \leq k \leq n$. The non-uniform expansion condition (P2) guarantees the existence of infinitely many hyperbolic times ν-almost everywhere. We refer the reader to Subsection 3.2 for more details. Let H denote the set of points with infinitely many hyperbolic times, $n_1(\cdot)$ be the first hyperbolic time map and $\Gamma_n = \{x \in M : n_1(x) > n\}$. We say that a probability measure η is *expanding* if $\eta(H) = 1$. In particular, any invariant expanding measure has only positive Lyapunov exponents. We also assume:

(P3) There is a unique equilibrium state μ for f with respect to ϕ, it is absolutely continuous with respect to ν with density $d\mu/d\nu \geq K^{-1}$ and $n_1 \in L^1(\mu)$.

The last assumption above essentially means that the decay of the first hyperbolic time map is at least polynomial of order $n^{-(1+\varepsilon)}$, for some $\varepsilon > 0$. We refer the reader to the works [ABV00, BS03, Yur03, Yur05, OV08, VV10], just to quote some classes of maps and potentials that satisfy our assumptions.

2.2.2. *Non-uniform specification property.* In contrast to the topological concept of specification we introduce a **measure theoretical** notion.

Definition 2.2. *We say that (f, μ) satisfy the* **non-uniform specification property** *if there exists $\delta > 0$ such that for μ-almost every x and every $0 < \varepsilon < \delta$ there exists an integer $p(x, n, \varepsilon) \geq 1$ satisfying*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} p(x, n, \varepsilon) = 0$$

and so that the following holds: given points $x_1, \dots, x_k$ in a full μ-measure set and positive integers $n_1, \dots, n_k$, if $p_i \geq p(x_i, n_i, \varepsilon)$ then there exists z that ε-shadows the orbits of each x_i during n_i iterates with a time lag of $p(x_i, n_i, \varepsilon)$ in between $f^{n_i}(x_i)$ and x_{i+1}, that is,

$$z \in B(x_1, n_1, \varepsilon) \quad \textit{and} \quad f^{n_1+p_1+\dots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$$

for every $2 \leq i \leq k$.

These notions means that almost every finite pieces of orbits are approximated by a real orbit such that the time lag between two consecutive pieces of orbits is small proportion of the size of the piece of orbit being shadowed. Clearly, if the strong specification property holds then (f, η) satisfies the non-uniform specification property for every f-invariant probability measure η. Let us also mention that

a notion of non-uniform specification property similar to the one introduced in [STV03] (using Pesin theory) would also be enough to obtain the lower bound estimates in Theorem 2.2 below. We shall not use or prove this fact here.

In opposition to the specification property we expect this weak form of specification to hold in a broad non-uniformly hyperbolic setting. We refer the reader to Section 6 for some examples in which the later condition holds but may fail to satisfy the specification property.

2.2.3. *Deviation bounds for non-uniformly expanding maps.* The following result exteds the large deviation results proven in [You90] for uniformly hyperbolic maps.

Theorem 2.2. *Let M be a compact manifold and $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside a critical or singular region $\mathcal{C}$ that satisfies (H) and (C). Let $\phi : M \setminus \mathcal{C} \to \mathbb{R}$ be an Hölder continuous potential and let ν and μ be probability measures given by (P1)-(P3). If $g \in C(M, \mathbb{R})$ and $c \in \mathbb{R}$ then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) \geq c \right) \\ \leq \max \left\{ \sup \left\{ -P + h_\eta(f) + \int \phi \, d\eta \right\}, \limsup_{n\to\infty} \frac{1}{n} \log \mu(\Gamma_n) \right\}$$

where the supremum is taken over all invariant probability measures η such that $\int g \, d\eta \geq c$. If, in addition, f satisfies the specification property or (f, μ) satisfies the non-uniform specification property then

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq \sup \left\{ -P + h_\eta(f) + \int \phi \, d\eta \right\},$$

where the supremum is taken over all invariant probability measures η such that $\eta(H) = 1$, $\int g \, d\eta > c$ and $n_1 \in L^1(\eta)$.

In consequence one can estimate the decay of the deviation set as follows:

Corollary 2.1. *Under the previous assumptions,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g \, d\mu \right| \geq c \right) \\ \leq \max \left\{ \sup \left\{ -P + h_\eta(f) + \int \phi \, d\eta \right\}, \limsup_{n\to\infty} \frac{1}{n} \log \mu(\Gamma_n) \right\}$$

where the supremum is taken over all invariant probability measures η such that $| \int g \, d\eta - \int g d\mu| \geq c$, and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g \, d\mu \right| > c \right) \\ \geq \sup \left\{ -P + h_\eta(f) + \int \phi \, d\eta \right\}$$

where the supremum is taken over all invariant probability measures η such that $\eta(H) = 1$, $| \int g \, d\eta - \int g d\mu| > c$ and $n_1 \in L^1(\eta)$.

Some comments are in order. First notice that if the first hyperbolic time map fails to have exponential decay then the right hand side in the previous upper bound is zero, since the other term is also non-positive. In [MN08] less than exponential deviations are proven for systems that admit a Young tower with inducing time

has polynomial decay. This reenforces that some condition on the decay of the first hyperbolic time as above should not be removable. See Example 6.2 for a more detailed discussion. Moreover, we expect Theorem 2.1 to hold in the more general setting of zooming measures introduced in [Pin09] since our ingredients are bounded distortion and growth to large scale.

3. Preliminary results

3.1. **Metric Entropy.** First we recall some definitions. Let $\varepsilon > 0$ and $n \geq 1$ be arbitrary. A set $E \subset M$ is (n, ε)-*separated* if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ with $x \neq y$, where $d_n : M \times M \to \mathbb{R}_0^+$ is the metric given by

$$d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y)).$$

If, in addition, E has maximal cardinality we say that it is a *maximal* (n, ε)-*separated* set. Note that for any maximal (n, ε)-separated set E, the dynamical balls $B(x, n, \varepsilon)$ centered at points in E are pairwise disjoint and that the union $\cup_{x \in E} B(x, n, 2\varepsilon)$ covers M. We recall some properties of topological and metric entropy.

Proposition 3.1. [Bow71] *Let $f : M \to M$ be a continuous map in a compact metric space M. If $h_{\text{top}}(f)$ denotes the topological entropy of f then*

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon),$$

where $N(n, \varepsilon)$ the minimum number of (n, ε) dynamical balls necessary to cover M.

A metric counterpart of this result is as follows. Let η be an invariant probability measure and $\delta > 0$ arbitrary. Given $\varepsilon > 0$ let $N(n, \varepsilon, \delta)$ be the minimum number of (n, ε)-dynamical balls necessary to cover a set of measure larger than $1 - \delta$.

Proposition 3.2. [Kat80, Theorem I.I] *Let $f : M \to M$ be a homeomorphism in a compact metric space M and η an f-invariant probability measure. Hence*

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta),$$

for every $\delta > 0$.

3.2. **Hyperbolic times.** In this subsection we recall some properties of hyperbolic times from [ABV00]. Given $\delta > 0$, let $\operatorname{dist}_\delta(x, \mathcal{C})$ be the δ-truncated distance from a point x to $\mathcal{C}$ defined as $\operatorname{dist}(x, \mathcal{C})$ if $\operatorname{dist}(x, \mathcal{C}) < \delta$ and equal to 1 otherwise.

Definition 3.1. *We say that (f, η) is* non-uniformly expanding *if there exists $N \geq 1$ and $\sigma > 1$ such that almost every x satisfies*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^N(f^{jN}(x)^{-1}\| \leq -2 \log \sigma < 0 \tag{3.1}$$

and the slow recurrence condition*: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for μ-almost every point $x \in M$ it holds that*

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log d_\delta(f^j(x), \mathcal{C}) < \varepsilon. \tag{3.2}$$

Let B, β be given by condition $(H2)$ and take $0 < b < \{\frac{1}{2}, \frac{1}{2\beta}\}$. A sufficiency criterium for the existence of hyperbolic times is given as application of Pliss' lemma.

Lemma 3.1. [ABV00, Lemma 5.4] *There exists constants $\theta > 0$ and $\delta > 0$ (depending only on f and c) such that if $x \in M \setminus \cup_n f^n(\mathcal{C})$ satisfies (3.1) and (3.2) then the following holds: for every large $N \geq 1$ there exist a sequence of integers $1 \leq n_1(x) < n_2(x) < \cdots < n_l(x) \leq N$, with $l \geq \theta n$ so that*

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x)^{-1}\| \leq \sigma^{-k} \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{C}) > \sigma^{bk}. \tag{3.3}$$

One of the main features of hyperbolic times is stated below.

Lemma 3.2. [ABV00, Lemma 2.7] *Given $c > 0$ and $\delta > 0$ there exists a constant $\delta_1 = \delta_1(c, \delta, f) > 0$ such that if n is a hyperbolic time for a point x then f^n maps diffeomorphically the dynamical ball $V_n(x) = B(x, n, \delta_1)$ onto $B(f^n(x), \delta_1)$ and*

$$d(f^{n-j}(y), f^{n-j}(z)) \leq \sigma^{-\frac{j}{2}} \, d(f^n(y), f^n(z))$$

for every $1 \leq j \leq n$ and every $y, z \in V_n(x)$.

Using that $J_\nu f = \lambda e^{-\phi}$ is Hölder continuous and the backward distances contraction at hyperbolic times we obtain a bounded distortion property.

Corollary 3.1. *There exists $K_0 > 0$ such that for every $y, z \in V_n(x)$*

$$K_0^{-1} \leq \frac{J_\nu f^n(y)}{J_\nu f^n(z)} \leq K_0.$$

3.3. **Control of the measure of dynamical balls.** Now we prove a useful lemma on the measure of dynamical balls for weak Gibbs measures. In what follows δ_1 stands for the diameter of the hyperbolic ball as in Lemma 3.2.

Lemma 3.3. *For every $0 < \varepsilon < \delta_1$ there exists a positive constant $K(\varepsilon) > 0$ such that if n is a hyperbolic time for x and $B(y, n, \varepsilon) \subset B(x, n, \delta)$ then*

$$K(\varepsilon)^{-1} \leq \frac{\nu(B(y, n, \varepsilon))}{e^{-Pn + S_n\phi(y)}} \leq K(\varepsilon),$$

where $P = \log \lambda$.

Proof One has $f^n(B(y, n, \varepsilon)) = B(f^n(y), \varepsilon)$ by backward distance contraction at hyperbolic times. Hence, Corollary 3.1 asserts that

$$1 \geq \nu(B(f^n(y), \varepsilon)) = \int_{B(y,n,\varepsilon)} e^{-S_n\psi} \, d\nu \geq K_0^{-1} e^{Pn - S_n\phi(y)} \, \nu(B(y, n, \varepsilon)).$$

Using that ν is positive on open sets and the compactness of M it follows that the measure of every ball of radius ε is bounded away from zero. Thus the other inequality is obtained analogously. □

The following very interesting consequence is that dynamical balls have comparable measure that only depends on the center of the ball.

Corollary 3.2. *Assume that $x \in H$. For every $0 < \varepsilon < \delta_1$ and $n \geq 1$ there exists a positive constant $K_n(x, \varepsilon) > 0$ such that if $B(y, n, \varepsilon) \subset B(x, n, \delta)$ then*

$$K_n(x, \varepsilon)^{-1} \leq \frac{\nu(B(y, n, \varepsilon))}{e^{-Pn + S_n\phi(y)}} \leq K_n(x, \varepsilon).$$

Proof Given an arbitrary n write $n_i(x) \le n < n_{i+1}(x)$, where n_i and n_{i+1} are consecutive hyperbolic times for x. Using that $B(y,n,\varepsilon) \subset B(y,n_i(x),\varepsilon)$ it is clear that

$$\begin{aligned}\nu(B(y,n,\varepsilon)) &\le K(\varepsilon)e^{(\sup|\phi|+|P|)(n-n_i(x))}\, e^{-Pn+S_n\phi(y)}\\ &\le K_n(x,\varepsilon)\, e^{-Pn+S_n\phi(y)},\end{aligned}$$

with $K_n(x,\varepsilon) = K(\varepsilon)\exp[(\sup|\phi|+|P|)(n-n_i(x))]$ (depends only on the center x). This finishes the proof of the corollary. □

Now we prove that the constants K_n have subexponential growth with respect to every invariant expanding measure such that the first hyperbolic time map is integrable. More precisely,

Lemma 3.4. *Let η be an f-invariant and expanding probability measure so that $n_1 \in L^1(\eta)$ and let $K_n(x,\varepsilon)$ be given as above. Then,*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log K_n(x,\varepsilon) = 0 \tag{3.4}$$

for η-almost every x. In consequence, $\psi = \phi - P$ belongs to $\mathcal{F}(H)$.

Proof This proof resembles the one of Proposition 3.8 in [OV08]. Let η be an f-invariant, expanding probability measure so that $n_1 \in L^1(\eta)$ and take $\beta > 0$ arbitrary. Given $x \in H$, $n \ge 1$ and $0 < \varepsilon < \delta_1$ recall that $K_n(x,\varepsilon) \le K(\varepsilon)\exp[(|P| + \sup|\phi|)n_1(f^{n_i(x)}(x))]$, where $n_i(x) \le n \le n_{i+1}(x)$ are consecutive hyperbolic times for x. Set $C_\beta = \beta n/(|P| + \sup|\phi|) - \log K(\varepsilon)$. If $K_n(x,\varepsilon) > e^{\beta n}$ this implies that $n_1(f^k(x)) > C_\beta n \ge C_\beta\, k$, where $k = n_i(x)$. This shows that

$$\begin{aligned}\{x\in H : K_n(x,\varepsilon) > e^{\beta n} \text{ i.o.}\} &\subset \{x \in H : n_1(f^n(x),\varepsilon) > e^{\beta n} \text{ i.o.}\}\\ &\subset \bigcup_{n\ge 1}\{x\in H : n_1(f^n(x)) > C_\beta n\}\end{aligned}$$

Furthermore, using the invariance of η and the integrability assumption

$$\sum_{n=1}^{+\infty}\eta\big(x\in H : n_1(f^n(x)) > C_\beta n\big) = \sum_{n=1}^{+\infty}\eta\big(x\in H : n_1(x) > C_\beta n\big) \le \int n_1\, d\eta < \infty.$$

Using Borel-Cantelli lemma this proves that $K_n(x,\varepsilon) \le e^{\beta n}$ for all but finitely many values of n for η-almost every x. Since β was taken arbitrary, this completes the proof of the first claim above.

Using that $n_1 \in L^1(\mu)$ and $d\mu/d\nu$ is bounded from below by a constant (recall assumption (P3)) it follows that (3.4) holds ν-almost everywhere. Together with Corollary 3.2 this shows that $\psi = \phi - P$ belongs to $\mathcal{F}(H)$ and that ν is a weak Gibbs measure. This finishes the proof of the lemma. □

Remark 3.1. *It follows from* (3.4) *and the definition of the constants $K_n(x,\varepsilon)$ that if $n_1 \in L^1(\eta)$ then given $\beta > 0$, for η-almost every x there exists $n_x \ge 1$ such that $n - n_i(x) \le \beta n$ for every $n \ge n_x$. In fact we prove even more: given $\beta > 0$ then for η-almost every x there exists $n_x \ge 1$ such that $n_{i+1}(x) - n_i(x) \le \beta n$ for every $n \ge n_x$.*

4. Abstract deviation bounds

In this section we prove Theorem 2.1. Upper and lower bounds for the measure of the deviation sets are given separately.

4.1. **Upper bound.** Let $g \in C(M, \mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$ be fixed. We want to prove that for every small $\varepsilon, \beta > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu \left[x \in M : \frac{1}{n} S_n g(x) \geq c \right] \leq \max \left\{ \delta(\varepsilon, \beta) \,, \sup \left\{ h_\eta(f) - \int \psi \, d\eta \right\} + \beta \right\}$$

where the supremum is taken over all invariant probability measures η such that $\int g \, d\eta \geq c$. We use the following result from Calculus (see e.g. [Wal82, Lemma 9.9]).

Lemma 4.1. *Given $n \geq 1$, real numbers $(a_i)_{i=1..n}$ and $0 \leq p_i \leq 1$ such that $\sum_{i=1}^n p_i = 1$ then*

$$\sum_{i=1}^n p_i (a_i - \log p_i) \leq \log \left(\sum_{i=1}^n e^{a_i} \right),$$

and the equality holds if and only if $p_i = \frac{e^{a_i}}{\sum_j e^{a_j}}$.

Let B_n denote the set of points $x \in M$ so that $S_n g(x) \geq cn$. Recall that Λ is a ν-full measure set and, for every $x \in \Lambda$ and every small $\varepsilon > 0$ it holds that

$$\nu(B(x, n, \varepsilon)) \leq K_n(x, \varepsilon) e^{-S_n \psi(x)},$$

with $\limsup_n \frac{1}{n} \log K_n(x, \varepsilon) = 0$. Let $\beta > 0$ and $0 < \varepsilon < \delta_0$ be arbitrary small and $n \geq 1$ be fixed. Then $B_n \subset \Delta_n^c \cup (B_n \cap \Delta_n)$, where Δ_n is as in (2.2). Moreover, if $E_n \subset B_n \cap \Delta_n$ is a maximal (n, ε)-separated set, $B_n \cap \Delta_n$ is contained in the union of the dynamical balls $B(x, n, 2\varepsilon)$ centered at points of E_n and, consequently,

$$\nu(B_n) < \nu(\Delta_n^c) + e^{\beta n} \sum_{x \in E_n} e^{-S_n \psi(x)} \tag{4.1}$$

for every n. Now, consider the probability measures σ_n and η_n given by

$$\sigma_n = \frac{1}{Z_n} \sum_{x \in E_n} e^{-S_n \psi(x)} \delta_x \quad \text{and} \quad \eta_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \sigma_n,$$

where $Z_n = \sum_{x \in E_n} e^{-S_n \psi(x)}$, and let η be an weak* accumulation point of the sequence $(\eta_n)_n$. It is not hard to check that η is an f-invariant probability measure. Assume $\mathcal{P}$ is a partition of M with diameter smaller than ε and $\eta(\partial\mathcal{P}) = 0$. Each element of $\mathcal{P}^{(n)}$ contains a unique point of E_n. By the previous lemma

$$H_{\sigma_n}(\mathcal{P}^{(n)}) - \int S_n \psi \, d\sigma_n = \log \Big(\sum_{x \in E_n} e^{-S_n \psi(x)} \Big)$$

which, as in the usual proof of the variational principle (see [Wal82, Pages 219-221]), guarantees that

$$\limsup_{n\to\infty} \frac{1}{n} \log Z_n \leq h_\eta(f) - \int \psi \, d\eta. \tag{4.2}$$

Observe also that $\int \psi \, d\eta \geq c$ by weak* convergence since E_n is contained in B_n and

$$\int g \, d\eta_n = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{Z_n} \sum_{x \in E_n} e^{S_n \phi(x)} . \, g \circ f^j(x) = \frac{1}{n} S_n g(x) \geq c.$$

Finally, it follows from (4.1) and (4.2) that for every $\beta > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu(B_n) \le \max\left\{\delta(\varepsilon,\beta)\ , h_\eta(f) - \int \psi\, d\eta + \beta\right\}$$
$$\le \max\left\{\delta(\varepsilon,\beta)\ , \sup\left\{h_\xi(f) - \int \psi\, d\xi\right\} + \beta\right\},$$

where the supremum is over all invariant probability measures. This completes the proof of the first statement in Theorem 2.1.

4.2. **Lower bound using ergodic measures.** Let $g : M \to \mathbb{R}$ be a continuous map, take $c \in \mathbb{R}$ and a $\beta > 0$ small. If η is an ergodic probability measure such that $\int g\, d\eta > c$ we claim that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c\right) \ge h_\eta(f) - h_\nu(f,\eta) - 2\beta.$$

Denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$ and fix $\delta_2 = \frac{1}{2}(\int g d\eta - c)$. Notice that $h_\eta(f) \le h_{\text{top}}(f) < \infty$ and that we may assume $h_\nu(f,\eta) < \infty$ (because otherwise there is nothing to prove). Hence η-almost every point x satisfies

$$h_\nu(f,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \nu(B(x,n,\varepsilon)) \le h_\nu(f,\eta) < \infty. \tag{4.3}$$

Since η is ergodic then $\frac{1}{n} S_n g(x) \to \int g\, d\eta$ for η-almost every x. Choose $\xi > 0$ by uniform continuity so that $|g(x) - g(y)| < \delta_2$ whenever $d(x,y) < \xi$. Observe that if $n_0 = n_0(\beta) \ge 1$ is large and $\delta \in (0,\xi)$ is small enough then the set D of points $x \in M$ satisfying

$$\frac{1}{n} S_n g(x) > c + \delta_2 \quad \text{and} \quad \nu(B(x,n,\varepsilon)) \ge e^{-[h_\nu(f,\eta)+\beta]n} \tag{4.4}$$

has η-measure at least $\frac{1}{2}$, and that the minimal number $N(n, 2\varepsilon, \frac{1}{2})$ of $(n, 2\varepsilon)$-dynamical balls necessary to cover a set of η-measure at least $\frac{1}{2}$ satisfies

$$N\left(n, 2\varepsilon, \frac{1}{2}\right) \ge e^{[h_\eta(f)-\beta]n} \tag{4.5}$$

for every $n \ge n_0$ and every $0 < \varepsilon \le \delta$. Indeed, the existence of such n_0 and δ is a consequence of Proposition 3.2, the definition of relative entropy and ergodicity. Moreover, it follows from our choice of ξ and δ_2 that

$$B_n \supset \bigcup_{x\in D} B(x,n,\varepsilon) \supset D$$

for all $n \ge n_0$ and $0 < \varepsilon < \xi$. So, if $\varepsilon > 0$ is small and $E_n \subset D$ is a maximal (n,ε)-separated set, using that the dynamical balls $B(x,n,\varepsilon)$ centered at points in E_n are pairwise disjoint contained in B_n and the union $\cup_{x\in E_n} B(x,n,2\varepsilon)$ covers D, relations (4.4) and (4.5) yield that

$$\nu(B_n) \ge \nu\Big(\bigcup_{x\in E_n} B(x,n,\varepsilon)\Big) \ge \sum_{x\in E_n} \nu\Big(B(x,n,\varepsilon)\Big) \ge e^{[h_\eta(f)-h_\nu(f,\eta)-2\beta]n}$$

whenever $n \ge n_0$, which proves our claim. The second second assertion in Theorem 2.1 follows from the arbitrariness of β.

Remark 4.1. *Since $B_n \supset B_n \cap \Delta_n$, where Δ_n is as in* (2.2) *then $\nu(B_n) \geq \nu(B_n \cap \Delta_n)$. However, we have no estimate whatsoever for the measure of the intersection in terms of $\nu(\Delta_n)$. Hence the previous result shows only that the measure of the points with predetermined Birkhoff averages decreases at most exponentially fast.*

4.3. **Lower bound over all invariant measures.** The proof of the last statement in Theorem 2.1 is divided in two steps. First we prove the lower bound when the supremum is restricted over ergodic measures. Afterwards we deduce the general bound using that every invariant measure can be approximated by a finite collection of ergodic measures and the specification to "glue" together finite pieces of orbits. We begin by the following lemma.

Lemma 4.2. *If $g \in C(M, \mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$ then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq h_\eta(f) - \int \psi \, d\eta$$

for every ergodic probability measure η such that $\eta(\Lambda) = 1$ and $\int g \, d\eta > c$.

Proof Fix $g \in C(M, \mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$, and denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$. Let $\beta > 0$ be a small constant and $\delta_2 = \frac{1}{2}(\int g d\eta - c)$. Let $\xi > 0$ be given by uniform continuity such that $|g(x) - g(y)| < \delta_2$ for any points $x, y \in M$ at distance smaller than ξ. As before, if n_0 is large enough and $0 < \varepsilon < \xi$ is small then the set D of points $x \in \Lambda$ satisfying

$$\frac{1}{n} S_n g(x) > c + \delta_2, \quad K_n(x, \varepsilon)^{-1} \geq e^{-\beta n} \quad \text{and} \quad \frac{1}{n} S_n \psi(x) < \int \psi \, d\eta + \beta \tag{4.6}$$

for every every $n \geq n_0$ has η-measure at least $\frac{1}{2}$ and $N\left(n, 2\varepsilon, \frac{1}{2}\right) \geq e^{(h_\eta(f) - \beta)n}$ for every $n \geq n_0$. Then, using that $B_n \supset \bigcup_{x \in D} B(x, n, \varepsilon) \supset D$ it follows that

$$\nu(B_n) \geq \sum_{x \in E_n} \nu \Big(B(x, n, \varepsilon) \Big) \geq e^{[h_\eta(f) - \int \psi \, d\eta - 3\beta] n}$$

for every maximal (n, ε)-separated set $E_n \subset D$. This proves that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq h_\eta(f) - \int \psi \, d\eta - 3\beta.$$

Since β was taken arbitrary the statement in the lemma follows directly. □

The following result asserts that any invariant probability measure can be approximated by a finite convex combination of ergodic measures supported in Λ.

Lemma 4.3. *Let $\eta = \int \eta_x d\eta(x)$ be the ergodic decomposition an f-invariant probability measure η such that $\eta(\Lambda) = 1$. Given $\beta > 0$ and a finite set $(\psi_j)_{1 \leq j \leq r} \subset C(M)$ of continuous functions, there are positive real numbers $(a_i)_{1 \leq i \leq k}$ satisfying $a_i \leq 1$ and $\sum a_i = 1$, and finitely many points $x_1, \dots, x_k$ such that the ergodic measures $\eta_i = \eta_{x_i}$ from the ergodic decomposition satisfy*

(i) $\eta_i(\Lambda) = 1$;
(ii) $h_{\hat{\eta}}(f) \geq h_\eta(f) - \beta$; *and*
(iii) $|\int \psi_j \, d\hat{\eta} - \int \psi_j \, d\eta| < \beta$ *for every* $1 \leq j \leq r$;

where $\hat{\eta} = \sum_{i=1}^{k} a_i \eta_i$.

Proof Fix the f-invariant probability measure η such that $\eta(\Lambda) = 1$. By ergodic decomposition theorem and convexity of the entropy, we can write $\eta = \int \eta_x \, d\eta(x)$ and $h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x)$, where each η_x denotes an ergodic component of η. Clearly $\eta_x(\Lambda) = 1$ for η-almost every x. Let $\mathcal{P}$ be a small finite partition of the space $\mathcal{M}(\Lambda)$ of invariant probability measures supported in Λ such that

$$\Big| \int \psi_j \, d\xi_1 - \int \psi_j \, d\xi_2 \Big| < \beta \tag{4.7}$$

for every $1 \le j \le e$ and every pair of probability measures ξ_1, ξ_2 in the same partition element. Set $k = \#\mathcal{P}$ and $a_i = \eta(P_i)$ for every element P_i in $\mathcal{P}$. For every $1 \le i \le k$ pick an ergodic measure $\eta_i = \eta_{x_i} \in P_i$ satisfying $h_{\eta_x}(f) \le h_{\eta_i}(f) + \beta$ for η-almost every $\eta_x \in P_i$. Part (i) in the lemma is immediate. On the other hand, (ii) follows because

$$h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x) \le \sum_{i=1}^{k} a_i \, h_{\eta_i}(f) + \beta = h_{\hat\eta}(f) + \beta.$$

Finally, (4.7) implies that

$$\Big| \int \psi_j \, d\eta - \int \psi_j \, d\hat\eta \Big| = \Big| \int \left(\int \psi_j \, d\eta_x \right) \, d\eta(x) - \sum_{i=1}^{k} a_i \int \psi_j \, d\eta_i \Big| \le \sum_{i=1}^{k} a_i \beta = \beta$$

for every j. This proves (iii) and finishes the proof of the lemma. $\square$

Now we will finish the proof of Theorem 2.1.

Proof [Proof of Theorem 2.1(continuation)]
Take $g \in C(M, \mathbb{R})$, $c \in \mathbb{R}$, $\psi \in \mathcal{F}(\Lambda)$ and let η be an invariant probability measure such that $\eta(\Lambda) = 1$ and $\int g \, d\eta > c$. Denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$. Take $\beta > 0$ arbitrary small, $\delta_2 = \frac{1}{5}(\int g\, d\eta - c)$ and the measure $\hat\eta = \sum_{i=1}^{k} a_i \eta_i$ given by Lemma 4.3 that satisfies

$$h_{\hat\eta}(f) \ge h_\eta(f) - \beta \,, \quad \int g \; d\hat\eta \ge \int g \; d\eta - \beta \quad \text{and} \quad \int \psi \; d\hat\eta \le \int \psi \; d\eta + \beta.$$

Since β is small we can assume $\int g\, d\hat\eta > c + 4\delta_2$. Now we claim that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \ge h_{\hat\eta}(f) - \int \psi \, d\hat\eta - 4\beta. \tag{$\star\star$}$$

As before, we may choose n_0 sufficiently large and δ small enough so that, for every $1 \le i \le k$, the set D_i of points $x \in \Lambda$ such that

$$\frac{1}{n} S_n g(x) > \int g \, d\eta_i - \beta, \quad \frac{1}{n} S_n \psi(x) < \int \psi \, d\eta_i + \beta \quad \text{and} \quad K_n(x, \varepsilon)^{-1} \ge e^{-\beta n}$$

for every every $n \ge n_0$ and $0 < \varepsilon \le \delta$ has η_i-measure at least $\frac{1}{2}$. Hence, given large n, small $\varepsilon > 0$ and $1 \le i \le k$ we proceed as in the proof of Lemma 4.2 to obtain a finite set $E_n^i \subset D_i$ so that

(1) E_n^i is a maximal $([a_i n], \varepsilon)$-separated set in D_i;
(2) $\# E_n^i \ge e^{(h_{\eta_i}(f) - \beta)\, [a_i n]}$; and
(3) for every $x \in E_n^i$ it holds

$$\frac{1}{[a_i n]} S_{[a_i n]} g(x) > \int g \, d\eta_i - \beta \quad \text{and} \quad \frac{1}{[a_i n]} S_{[a_i n]} \psi(x) < \int \psi \, d\eta_i + \beta.$$

By the specification property, for every sequence $(x_1, x_2, \dots, x_k)$ with $x_i \in E_n^i$ there exists $x \in M$ that ε-shadows each x_i during $[a_i n]$ iterates with a time lag of $N(\varepsilon)$ iterates in between. Consequently, if n is large and $\tilde{n} = \sum_i [a_i n] + kN(\varepsilon)$ then $S_{\tilde{n}} g(x) > (c + 2\delta_2)\tilde{n}$. Since the dynamical ball $B(x, \tilde{n}, \varepsilon/8)$ is contained in $B_{\tilde{n}} \cap B(x_1, \tilde{n}, \delta_0)$ for every large n, it follows from (2.1) that

$$\nu\Big(B(x, \tilde{n}, \varepsilon)\Big) \geq K_{\tilde{n}}(x_1, \varepsilon)^{-1}\ e^{-S_{\tilde{n}}\psi(x)} \geq e^{-\beta\tilde{n}}\ e^{-(\int \psi\, d\eta_i + 2\beta)\tilde{n}}.$$

On the other hand, there are at least $\# E_n^1 \times \cdots \times \# E_n^k$ such pairwise disjoint dynamical balls contained in $B_{\tilde{n}}$. It follows that

$$\nu(B_{\tilde{n}}) \geq \sum_x \nu(B(x, \tilde{n}, \varepsilon)) \geq e^{[h_{\hat{\eta}}(f) - \int \psi d\hat{\eta} - 4\beta]\tilde{n}}$$

for every large n, which gives $(\star\star)$. Since β was chosen arbitrarily small then

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq h_\eta(f) - \int \psi\, d\eta - 6\beta,$$

which proves the third part in Theorem 2.1 and finishes its proof. □

5. Deviation estimates for non-uniformly expanding maps

In this section we use some of the ideas involved in the proof of Theorem 2.1 together with the key notion of non-uniform specification to prove the large deviation bounds in Theorem 2.2. Through the section, let M be a compact manifold and $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside a critical/singular region $\mathcal{C}$ that satisfies (H). Let $\phi : M \setminus \mathcal{C} \to \mathbb{R}$ be an Hölder continuous potential such that (P1)-(P3) hold. Denote by ν the corresponding weak Gibbs measure and by μ the unique equilibrium state for f with respect to ϕ.

5.1. **Upper bound.** In this subsection we obtain an upper bound for the measure of the deviation set of non-uniformly expanding maps.

Lemma 5.1. *If $g \in C(M, \mathbb{R})$ and $c \in \mathbb{R}$ then it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) \geq c \right) \\ \leq \max \left\{ \sup \left\{ -P + h_\eta(f) + \int \phi\, d\eta \right\}, \limsup_{n\to\infty} \frac{1}{n} \log \mu(\Gamma_n) \right\},$$

where the supremum is taken over all f-invariant measures η such that $\int g\, d\eta \geq c$.

Proof Let $\beta > 0$ be given. First we observe that the computations in Lemma 3.4 show that $\psi = \phi - P \in \mathcal{F}(H)$ and that there exists $C_\beta > 0$ such that the set Δ_n as in (2.2) is given by

$$\Delta_n \supset \{x \in H : n_1(f^{n_i(x)}(x)) \leq C_\beta n\}$$

where $n_i(x) \leq n \leq n_{i+1}(x)$ are consecutive hyperbolic times for x. In particular, using that $d\mu/d\nu \geq K^{-1}$, computations analogous to the ones in the proof of

Lemma 3.4 also give that

$$
\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\nu(\Delta_n^c) &\le \limsup_{n\to\infty}\frac{1}{n}\log\nu\left[\bigcup_{1\le k\le n}\{x\in H\colon n_i(x)=k,\, n_1(f^k(x))>C_\beta n\}\right]\\
&\le \limsup_{n\to\infty}\frac{1}{n}\log\big(Kn\mu(x\in H: n_1(x)>C_\beta n)\big)\\
&\le \limsup_{n\to\infty}\frac{1}{n}\log\big(KC_\beta^{-1}n\mu(x\in H: n_1(x)>n)\big)\\
&= \limsup_{n\to\infty}\frac{1}{n}\log\mu\left(\Gamma_n\right).
\end{aligned}
$$

Hence $\delta(\varepsilon,\beta)$ does not depend neither on ε or β. Thus, the lemma is an immediate consequence of the first part in Theorem 2.1. $\square$

5.2. **Lower bound estimates.** Here we make use of approximation Lemma 4.3 and specification properties to obtain lower bounds for the deviation sets. We are now in a position to prove the following:

Proposition 5.1. *Assume that $g\in C(M,\mathbb{R})$ and $c\in\mathbb{R}$. If either f satisfies the specification property or (f,μ) satisfies the non-uniform specification property then*

$$
\liminf_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M: \frac{1}{n}S_ng(x)>c\right)\ge -P+h_\eta(f)+\int\phi\,d\eta,
$$

for every invariant and expanding *probability measure η satisfying $\int g\,d\eta>c$ and $n_1\in L^1(\eta)$.*

Proof Note that $\psi=\phi-P\in\mathcal{F}(H)$ by Lemma 3.4. Set $g\in C(M,\mathbb{R})$ and $c\in\mathbb{R}$, and let B_n be the set of points $x\in H$ such that $S_ng(x)>cn$. Fix $\beta>0$ arbitrary small and let η be an f-invariant and expanding probability measure such that $\int g\,d\eta>c$ and $n_1\in L^1(\eta)$. Set also $\delta_2=\frac{1}{5}(\int g\,d\eta-c)$. Observe that almost every ergodic component η_x of the invariant measure η satisfy $n_1\in L^1(\eta_x)$. It follows from Lemma 4.3 that there are exists a probability vector $(a_1,\dots,a_k)$ and f-invariant ergodic probability measures $(\eta_i)_{1\le j\le k}$ such that $\hat\eta=\sum a_j\eta_j$ satisfies

$$
h_{\hat\eta}(f)\ge h_\eta(f)-\beta\ ,\quad \int g\ d\hat\eta\ge\int g\ d\eta-\beta\quad\text{and}\quad \int\psi\ d\hat\eta\le\int\psi\ d\eta+\beta.
$$

Moreover, it is not hard to check that we can assume $n_1\in L^1(\eta_j)$ for every $1\le j\le k$. So, the Ergodic Theorem and Remark 3.1 guarantee that one can pick $n_0\ge1$ large and δ small enough such that, for every $1\le j\le k$, the set D_j of points $x\in H$ such that

$$
n_{i+1}(x)-n_i(x)\le\beta n,\quad \frac{1}{n}S_ng(x)>\int g\,d\eta_j-\beta\quad\text{and}\quad\frac{1}{n}S_n\psi(x)<\int\psi\,d\eta_j+\beta,
$$

for every $n\ge n_0$, has η_j-measure larger than $\frac{1}{2}$. Recall that $n_i(x)\le n<n_{i+1}(x)$ are consecutive hyperbolic times for x. If $0<\varepsilon\ll\delta_1$ (as in Lemma 3.2) is small then $|g(x)-g(y)|<\delta_2$ whenever $|x-y|<\varepsilon$. As in Subsection 4.2, for every large n and small $\varepsilon>0$ there exists a set $E_n^j\subset D_j$ such that

(1) E_n^j is a maximal $([a_jn],\varepsilon)$-separated set;
(2) $\#E_n^j\ge e^{(h_{\eta_j}(f)-\beta)[a_jn]}$;

(3) for every $x \in E_n^j$ it holds

$$\frac{1}{[a_j n]} S_{[a_j n]} g(x) > \int g \, d\eta_j - \beta \quad \text{and} \quad \frac{1}{[a_j n]} S_{[a_j n]} \psi(x) < \int \psi \, d\eta_j + \beta.$$

Condition (1) yield that the dynamical balls $B(x, [a_j n], \varepsilon)$ centered at points in E_n^j are pairwise disjoint. We divide the remaining of the proof in two cases:

First case: f satisfies the specification property

Given any sequence $(z_1, z_2, \dots, z_k)$ with $z_j \in E_n^j$ there exists some point $z \in M$ that ε-shadows each z_j during $\ell_j := [a_j n]$ iterates with a time lag of $p_j = N(\varepsilon)$ iterates as in Definition 2.1. Let $n_j := n_i(z_j)$ denote the last hyperbolic time for z_j smaller than ℓ_j and write $\ell_j = n_j + t_j$ for some $t_j \geq 0$.

Therefore, if we set $p_k = 0$ and take $\tilde n = \sum_{j=1}^k \big(\ell_j + p_j\big)$ one can use that $\max\{k, t_k\} \leq \beta \ell_k \ll \beta(\tilde n - t_k)$ to deduce that

$$\begin{aligned}
S_{\tilde n} g(z) &\geq \sum_{j=1}^k S_{\ell_j} \, g(z_j) - \delta_2 \sum_{j=1}^k \ell_j - \sup |g| \, k \, N(\varepsilon) \\
&\geq \sum_{j=1}^k \Big(\int g \, d\eta_j - \delta_2 - \beta \Big) \, \ell_j - \sup |g| \, k \, N(\varepsilon) \\
&> (c + 3\delta_2) n - \sup |g| \, k \, N(\varepsilon) \\
&\geq (c + 3\delta_2) \tilde n - 2 \sup |g| \, k \, N(\varepsilon) \\
&\geq (c + 2\delta_2) \tilde n
\end{aligned}$$

provided that n is large enough. Hence $B(z, \tilde n, \varepsilon) \subset B_{\tilde n}$ and there are at least $e^{(h_{\hat\eta}(f) - 2\beta)\tilde n}$ such distinct dynamical balls. Since each of the points x_i were chosen in a full measure set then the weak Gibbs property yields an estimate for the measure of the corresponding dynamical balls. However, no a priori estimates on the measure of the specified orbit z is guaranteed. We claim that

$$\begin{aligned}
\nu(B(z, \tilde n, \varepsilon)) &\geq e^{-2k \sup |\psi| \, \beta \tilde n} \left(\prod_{j=1}^k e^{-S_{\ell_i} \psi(z_j)} \right) \\
&\geq e^{-2k \sup |\psi| \, \beta \tilde n} \, e^{-2\beta \tilde n} \, e^{-\tilde n \, \int \psi \, d\eta} \qquad (5.1)
\end{aligned}$$

for every large n.

Proof [Proof of the claim:] Let L and γ be given by condition (C). For notational simplicity set $\tilde z_{k+1} = f^{\sum_{j=1}^k (\ell_j + p_j)}(z) \in B(z_{k+1}, \ell_{k+1}, \varepsilon)$. Since $B(z, \tilde n, \varepsilon)$ contains $B(z, \tilde n + n_{k+1}, \varepsilon)$, where $\ell_k + n_{k+1}$ denotes the first hyperbolic time for z_k larger than ℓ_k, first we show that

$$f^{\tilde n + n_{k+1}}(B(z, \tilde n + n_{k+1}, \varepsilon)) = B(f^{\tilde n + n_{k+1}}(z), \varepsilon). \qquad (5.2)$$

Since $\ell_k + n_{k+1}$ is a hyperbolic time for z_k and $\varepsilon \ll \delta_1$ then there exists backward distance contraction and $f^{\ell_k + n_{k+1}}(B(\tilde z_k, \ell_k + n_{k+1}, 2\varepsilon)) = B(f^{\ell_k + n_{k+1}}(\tilde z_k), 2\varepsilon)$. In fact, using $d(f^{n_k}(\tilde z_k), f^{n_k}(z_k)) < \varepsilon$, that β is fixed arbitrary small and $\ell_k - n_k < \beta n$ (recall the definition of the set D_k) then

$$\operatorname{diam}(B(\tilde z_k, \ell_k + n_{k+1}, \varepsilon)) \leq \operatorname{diam}(B(\tilde z_k, n_k, \varepsilon)) \leq \varepsilon \sigma^{-\frac{1}{2} n_k} \leq \varepsilon \sigma^{-\frac{1}{2}[a_k - \beta] n} \ll \varepsilon$$

provided that n is large enough. Again, if n is large, using property (C) on the diameter of preimages this yields that there exists $\tilde{L} > 0$ so that

$$\begin{aligned} \operatorname{diam}(f^{-p_{k-1}-t_{k-1}}(B(\tilde{z}_k, \ell_k + n_{k+1}, \varepsilon))) &\leq \tilde{L}[\varepsilon \sigma^{-\frac{1}{2}[a_k - \beta]n}]^{\gamma^{p_{k-1}+t_{k-1}}} \\ &\leq \tilde{L}[\varepsilon \sigma^{-\frac{1}{2}[a_k - \beta]n}]^{\gamma^{N(\varepsilon)+\beta n}} \ll \varepsilon \end{aligned}$$

and, consequently, $f^{-p_{k-1}-t_{k-1}}(B(\tilde{z}_k, \ell_k + n_{k+1}, \varepsilon)) \subset B(f^{n_{k-1}}(\tilde{z}_{k-1}), 2\varepsilon)$. Recall that $\tilde{z}_k = f^{\ell_{k-1}+p_{k-1}}(\tilde{z}_{k-1})$, that n_{k-1} is a hyperbolic time for z_{k-1} and there exists backward distance contraction in the dynamical ball of radius $\delta_1 \gg 2\varepsilon$. Hence $f^{n_{k-1}}(B(\tilde{z}_{k-1}, n_{k-1}, 2\varepsilon)) = B(f^{n_{k-1}}(\tilde{z}_{k-1}), 2\varepsilon)$ and

$$\begin{aligned} &B(\tilde{z}_{k-1}, \ell_{k-1} + p_{k-1} + \ell_k + n_{k+1}, \varepsilon) \\ &\qquad = B(\tilde{z}_{k-1}, n_{k-1}, \varepsilon) \cap f^{-\ell_{k-1}-p_{k-1}}(B(\tilde{z}_k, \ell_k + n_{k+1}, \varepsilon)) \\ &\qquad = B(\tilde{z}_{k-1}, n_{k-1}, \varepsilon) \cap f^{-n_{k-1}}[f^{-p_{k-1}-t_{k-1}}(B(\tilde{z}_k, \ell_k + n_{k+1}, \varepsilon))] \\ &\qquad \subset B(\tilde{z}_{k-1}, n_{k-1}, \varepsilon) \cap f^{-n_{k-1}}[B(f^{n_{k-1}}(\tilde{z}_{k-1}), 2\varepsilon)]. \end{aligned}$$

Consequently, the dynamical ball $B(\tilde{z}_{k-1}, \ell_{k-1} + p_{k-1} + \ell_k + n_{k+1}, \varepsilon)$ is mapped diffeomorphically by $f^{\ell_{k-1}+p_{k-1}+\ell_k+n_{k+1}}$ onto the ball centered at $f^{\ell_k+n_{k+1}}(\tilde{z}_k)$ with radius ε. Using the same argument as above recursively we obtain (5.2) as desired.

It remains to compute the measure of $B(z, \tilde{n} + n_{k+1}, \varepsilon)$. Using (5.2), the fact that ν is a conformal measure with Jacobian $J_\nu f = e^{-\psi}$ and the bounded distortion property at hyperbolic times (see Corollary 3.1) it follows that

$$\begin{aligned} \nu(B(f^{\tilde{n}+n_{k+1}}(z),\varepsilon)) &= \int_{B(z,\tilde{n}+n_{k+1},\varepsilon)} e^{S_{\tilde{n}+n_{k+1}}\psi(y)}\, d\nu(y) \\ &\leq K_0^k e^{\sup|\psi|\,[n_{k+1}+\sum_j (t_j+p_j)]} \left(\prod_{j=1}^k e^{S_{\ell_i}\psi(z_j)} \right) \nu(B(z, \tilde{n} + n_{k+1}, \varepsilon)). \end{aligned}$$

Since ν is an open measure then every ball of radius ε has measure at least $C_\varepsilon > 0$. Finally, using that $n_k \leq \ell_k \leq (\ell_k + n_{k+1})$ are consecutive hyperbolic times for z_k then $n_{k+1} \leq (\ell_k + n_{k+1}) - n_k \leq \beta \ell_k = \beta[a_k n] \ll \beta \tilde{n}$ and so

$$\begin{aligned} \nu(B(z, \tilde{n}, \varepsilon)) &\geq \nu(B(z, \tilde{n} + n_{k+1}, \varepsilon)) \\ &\geq C_\varepsilon K_0^{-k} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right) e^{-\sup|\psi|\,[kN(\varepsilon)+(k+1)\beta\tilde{n}]} \\ &\geq e^{-2k\sup|\psi|\,\beta\tilde{n}} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right) \\ &\geq e^{-2k\sup|\psi|\,\beta\tilde{n}}\; \exp\Big(\sum_{j=1}^k (-\int \psi\, d\eta_j - \beta)\ell_j \Big) \\ &\geq e^{-2k\sup|\psi|\,\beta\tilde{n}}\, e^{-2\beta\tilde{n}}\, e^{-\tilde{n}\int \psi\, d\eta} \end{aligned}$$

for every large n, which proves our claim. □

We are now in a position to finish the proof of the first case of the proposition. Indeed, note that we obtain as a direct consequence of equation (5.1) that $\log \nu(B_{\tilde{n}}) \geq \Big(h_\eta(f) - \int \psi d\eta - 5\beta - 2\beta k \sup|\psi|\Big)\tilde{n}$ for every large n. Since β was

arbitrary this shows that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq h_\eta(f) - \int \psi \, d\eta.$$

Second case: (f, μ) satisfies the non-uniform specification property

In the case that (f, μ) satisfies the non-uniform specification property the computations are similar to the previous ones with the difference that the time lags given by non-uniform specification may be unbounded. Take n_0 large and δ small so that

$$p(x, n, \varepsilon) \leq \beta n$$

for every $x \in D_i$, $0 < \varepsilon \leq \delta$, $1 \leq i \leq k$ and $n \geq n_0$. Through the remaining of the proof set also $p_j := \max_{x \in E_n^j} p(x, n, \varepsilon)$.

For every sequence $(z_1, z_2, \ldots, z_k)$ with $z_j \in E_n^j$ there exists some point $z \in M$ that ε-shadows, in the non-uniform metric, each z_i during $\ell_j := [a_j n]$ iterates with a time lag of p_j iterates. Moreover, if $\tilde n = \sum_{j=1}^k \big(\ell_j + p_j\big)$, the set of points z obtained as above are $(\tilde n, \varepsilon)$ separated and there are at least $e^{(h_{\hat\eta}(f) - 2\beta)\tilde n}$ such points. Since $\beta > 0$ is small, observe that

$$S_{\tilde n} g(z) \geq \sum_{j=1}^k S_{\ell_j}\, g(z_j) - \delta_2 \sum_{j=1}^k \ell_j - \sup|g| \sum_{j=1}^k p_j, \tag{5.3}$$

which is bounded from below by

$$\sum_{j=1}^k \Big(\int g \, d\eta_j - 2\beta - \delta_2 \Big)\, \ell_j > \Big(\int g \, d\eta - 2\delta_2 \Big) n > (c + 2\delta_2) n + (c + \delta_2) \frac{\beta n}{\sup|g|} > (c + \delta_2)\tilde n$$

for every large n. It follows from our choice of ε that $B(z, \tilde n, \varepsilon) \subset B_{\tilde n}$. Given z as above, the computations involved in the proof of (5.1) give that

$$\begin{aligned} \nu(B(z, \tilde n + n_{k+1}, \varepsilon)) &\geq C(\varepsilon)^{-1} K_0^{-k} e^{-\sup|\psi|\, [n_{k+1} + \sum_j (t_j + p_j)]} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right) \\ &\geq e^{-3(k+1)\sup|\psi|\,\beta \tilde n} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right) \end{aligned}$$

and, consequently, $\nu(B(z, \tilde n, \varepsilon)) \geq e^{-3(k+1)\sup|\psi|\,\beta\tilde n}\, e^{-2\beta\tilde n}\, e^{-\tilde n \int \psi \, d\eta}$ for every large integer n. Henceforth, $\log \nu(B_{\tilde n}) \geq \Big(h_\eta(f) - \int \psi d\eta - 5\beta - 3(k+1)\beta \sup|\psi| \Big) \tilde n$ for large n. Since both $0 < \varepsilon < \delta$ and $\beta > 0$ were arbitrarily small and $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{p(x,n,\varepsilon)}{n} = 0$ for almost every x one obtains

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq h_\eta(f) - \int \psi \, d\eta.$$

The proof of the proposition is now complete. $\square$

6. Some applications

6.1. **One-dimensional examples.** Large deviations estimates for one-dimensional non-uniformly expanding maps were obtained only by Keller and Nowicki [KN92] for quadratic maps satisfying the Collet-Eckmann condition and by Araújo and Pacífico [AP06] to non-uniformly expanding quadratic maps. The first authors proved a large deviations principle for observables of bounded variation and the second authors obtained upper bounds for the measure of the deviation sets of any continuous observable. Using that every topologically mixing and continuous interval map satisfies specification (see [Blo83]) we will now discuss applications of our results to some important classes of examples.

Example 6.1. (Non-uniformly expanding quadratic maps)
We consider the class of quadratic maps f_a on the interval $[0,1]$ *given by*

$$f_a(x) = 1 - ax^2.$$

In [BC85]*, Benedicks and Carleson proved the existence of a positive Lebesgue measure set of parameters $\Omega \in [0,2]$ such that for every $a \in \Omega$ the quadratic map f_a has positive Lyapunov exponent and an unique absolutely continuous invariant probability measure μ_a. In fact, these maps are topologically mixing and $d\mu_a/dLeb \in L^p$ for every $p < 2$. It follows from the previous discussion that each f_a satisfies the specification property. Moreover, the same argument used in* [AP09] *to deal with infinitely many critical points is enough to guarantee that $Leb(\Gamma_n)$ decays exponentially fast (cf.* [AP06, Section 2.1]*).*

Now we notice that all invariant measures are expanding. Indeed, on the one hand [BK98, Proposition 3.1] *establishes for S-unimodal maps and any invariant measure $\lambda(\mu) \geq \lambda_{per}$, where $\lambda(\mu) = \int \log|f'|\,d\mu$ is the integrated Lyapunov exponent of μ and λ_{per} is the infimum of Lyapunov exponents among periodic orbits. On the other hand, it follows from* [NS98] *that the Collet-Eckmann is equivalent to $\lambda_{per} > 0$.*

Since there exist many invariant probability measures with integrable first hyperbolic time map we proceed to show that the measure of the deviation sets is exponential. Using that $d\mu_a/dLeb \in L^p$ for any $p \in (1,2)$ and that $Leb(\Gamma_n)$ decreases exponentially fast then, if $q > 1$ is satisfies $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality

$$\mu_a(\Gamma_n) = \int 1_{\Gamma_n} d\mu_a = \int 1_{\Gamma_n} \frac{d\mu_a}{d\nu} d\nu \leq \left\| \frac{d\mu_a}{d\nu} \right\|_p Leb(\Gamma_n)^q$$

also decreases exponentially fast and $n_1 \in L^1(\mu_a)$. Since μ_a is an equilibrium state for $\phi_a = -\log|f_a'|$ then it follows from Ruelle-Pesin's formulas that $P = 0$. Moreover, $\nu = Leb$ is an expanding conformal measure and so

$$\limsup_{n\to\infty} \frac{1}{n} \log Leb\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu_a \right| \geq c \right) \leq -\alpha$$

where

$$\alpha = \min\left\{ -\lim_{n\to\infty} \frac{1}{n} \log \mu_a(\Gamma_n), \sup\{-h_\eta(f) + \int \log|f_a'|\, d\eta\} \right\} > 0,$$

and the supremum in the right hand term is over all invariant measures η such that $|\int g d\eta - \int g d\mu_a| \geq c$. Analogously,

$$\liminf_{n\to\infty} \frac{1}{n} \log Leb\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu_a \right| > c \right) \geq -\beta$$

where $0 < \beta = \sup\{-h_\eta(f) + \int \log|f_a'|\, d\eta\}$ *and the supremum is taken over all invariant measures* η *such that* $n_1 \in L^1(\eta)$ *and* $|\int g d\eta - \int g d\mu_a| > c$*.*

The following example illustrates that the condition on the decay of the first hyperbolic time map is not removable from the upper bound.

Example 6.2. (Intermittency phenomena)
Given $\alpha \in (0,1)$*, let* $f : [0,1] \to [0,1]$ *be the* $C^{1+\alpha}$ *transformation of the interval given by*

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

known as the Maneville-Pomeau map. This transformation has 0 *as an indifferent fixed point (that is* $Df(0) = 1$*) and expansion everywhere else. the map presents an intermittency phenomenon. It is known that* f *has a finite absolutely continuous invariant probability measure* μ *with polynomial decay of correlations of order* $\mathcal{O}(n^{\frac{1}{\alpha}-1})$*. In fact that is also the decay of the tail of the first hyperbolic time with respect to* $m = Leb$*.*

By Ruelle-Pesin's formula, μ *is an equilibrium state for* f *with respect to the potential* $\phi = -\log|Df|$ *with pressure* $P := P(\phi) = 0$*. In fact,* μ *and* δ_0 *are the unique ergodic equilibrium states for* ϕ*. In addition, it is proved in* [Hu04] *that* $d\mu/dm \approx x^{-\alpha}$*. However,* $n_1 \notin L^1(\mu)$*. Roughly, partitioning the unit interval according to the sequence* $(\frac{1}{n})_n$ *it follows that*

$$\begin{aligned} \int n_1 d\mu &\ge \sum_{n\ge1} n_1(\frac{1}{n+1})\mu([\frac{1}{n+1},\frac{1}{n}]) \approx \sum_{n\ge1} n_1(\frac{1}{n+1})m([\frac{1}{n+1},\frac{1}{n}])(\frac{1}{n})^{-\alpha} \\ &= \sum_{n\ge1} n^\alpha\, n_1(\frac{1}{n+1})m([\frac{1}{n+1},\frac{1}{n}]), \end{aligned}$$

which is infinite because $n_1 \in L^1(m)$*. In consequence the Lebesgue measure of deviation sets decrease polynomially and* $\limsup \frac{1}{n}\log\mu(\Gamma_n) = 0$*. Since* f *admits a finite Markov partition then it satisfies the specification property. Therefore, it follows from Theorem 2.1 that for every* continuous *observable* g

$$\liminf_{n\to\infty} \frac{1}{n}\log Leb\left[x \in M : \left|\frac{1}{n}S_n g(x) - \int g\, d\mu\right| > c\right] \ge \sup_\eta \left\{h_\eta(f) - \int \log|Df|\, d\eta\right\} \tag{6.1}$$

where η *denotes an invariant measure so that* $\eta(H) = 1$*,* $|\int g\, d\eta - \int g\, d\mu| > c$ *and* $n_1 \in L^1(\eta)$*. If, in addition,* $g(0) = \int g\, d\mu$*, then* η *does not belong to the convex hull generated by the equilibrium states Leb and* δ_0*. Moreover, the supremum is taken over expanding measures and the right hand side of* (6.1) *is strictly negative, which proves that the measure of deviation sets decrease at most exponentially fast. We point out that polynomial upper and lower bounds for an open and dense set of* Hölder continuous *observables have been established in* [MN08, Mel09].

6.2. **Higher dimensional examples.** The next class of examples are multidimensional local diffeomorphisms obtained by local bifurcation of expanding maps and were introduced in [ABV00]. Although the original expanding maps satisfy the specification property we point out that the same should not hold for the perturbations.

Example 6.3. *Let f_0 be an expanding map in $\mathbb{T}^n$ and take a periodic point p for f_0. Let f be a C^1-local diffeomorphism obtained from f_0 by a bifurcation in a small neighborhood U of p in such a way that:*

1. *every point $x \in M$ has some preimage outside U;*
2. *$\|Df(x)^{-1}\| \leq \sigma^{-1}$ for every $x \in M \backslash U$, and $\|Df(x)^{-1}\| \leq L$ for every $x \in M$ where $\sigma > 1$ is large enough or $L > 0$ is sufficiently close to 1;*
3. *f is topologically exact: for every open set U there is $N \geq 1$ for which $f^N(U) = M$*

It follows from [VV10] *that f has a unique (ergodic) equilibrium state μ for the Hölder continuous potential $\phi = -\log|\det Df|$, it is absolutely continuous with respect to the conformal measure $\nu = Leb$ with density bounded away from zero and infinity, and it is expanding. We note also that that the equilibrium state μ also satisfies the non-uniform specification property.*

Lemma 6.1. *(f, μ) satisfies the non-uniform specification property.*

Proof First we note that since M is compact and f is topologically exact then for every $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that $f^{N_\varepsilon}(B) = M$ for every ball B of radius ε. Indeed, for every x let $N(x, \varepsilon) \geq 1$ be the minimum integer such that $f^{N(x,\varepsilon)}(B(x, \varepsilon/3)) = M$. By compacteness the open cover $(B(x, \varepsilon/3))_{x \in M}$ admits a finite covering $(B(x_i, \varepsilon/3))_{i=1..n}$. Hence, if $N_\varepsilon = \max\{N(x_i, \varepsilon) : i = 1..n\}$ then any ball B of radius ε contains a ball $B(x_j, \varepsilon/3)$, for some j, and so $f^{N_\varepsilon}(B) = M$.

It follows from [VV10] *that the equilibrium state μ is absolutely continuous with respect to a conformal measure ν with density bounded away from zero and infinity and $n_1 \in L^1(\mu)$. Moreover, the sequence $n_k(\cdot)$ of hyperbolic times is non-lacunar, that is $\frac{n_{k+1}-n_k}{n_k} \to 0$ at almost every x. Therefore, if $0 < \varepsilon < \delta$, n is large and $n_k(x) < n < n_{k+1}(x)$ are consecutive hyperbolic times then clearly $B(x, n_{k+1}, \varepsilon) \subset B(x, n, \varepsilon)$ and*

$$f^{n_{k+1}+N_\varepsilon}(B(x, n_{k+1}, \varepsilon)) = f^{N_\varepsilon}(B(f^{n_{k+1}}(x), \varepsilon)) = M.$$

Thus for any given $y \in M$ and proximity $\zeta > 0$ there exists $z \in B(x, n, \varepsilon)$ so that $f^{N_\varepsilon + n_{k+1}(x) - n}(f^n(z)) = f^{N_\varepsilon + n_{k+1}(x)}(z) \in B(y, \zeta)$. Take $p(x, n, \varepsilon) = N_\varepsilon + n_{k+1}(x) - n$. Then for any $x_1, \ldots, x_m$ in a full μ-measure set, any positive integers $k_1, \ldots, k_m$ and $p_i \geq p(x_i, n_i, \varepsilon)$ there exists $z \in M$ such that $z \in B(x_1, n_1, \varepsilon)$ and $f^{n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$ for every $2 \leq i \leq k$. To obtain the non-uniform specification property just note that

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{p(x, n, \varepsilon)}{n} \leq \lim_{\varepsilon \to 0} \limsup_{k \to \infty} \frac{N_\varepsilon + n_{k+1}(x) - n_k(x)}{n_k(x)} = 0.$$

□

Using the non-uniform specification property it is a consequence of Theorem 2.2 that

$$\limsup_{n \to \infty} \frac{1}{n} \log Leb\left(x \in M : \left|\frac{1}{n} S_n g(x) - \int g \, d\mu\right| \geq c\right)$$
$$\leq \max\left\{\sup\left\{-P + h_\eta(f) + \int \phi \, d\eta\right\}, \limsup_{n \to \infty} \frac{1}{n} \log \mu(\Gamma_n)\right\},$$

where the supremum taken over all invariant probability measures η satisfying $|\int g\, d\eta - \int g\, d\mu| > c$, and also

$$\liminf_{n\to\infty} \frac{1}{n} \log Leb \left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu \right| > c \right)$$
$$\geq \sup \left\{ -P + h_\eta(f) + \int \phi\, d\eta \right\},$$

where the supremum is taken over expanding f-invariant probability measures η such that $n_1 \in L^1(\eta)$ and $|\int g\, d\eta - \int g\, d\mu| > c$. Note that both rates are exponential.

We also prove that a robust class of multidimensional non-uniformly expanding maps with singularities also satisfy this weak form of specification.

Example 6.4. (Viana maps)
In [Via97]*, the author introduced a robust class of multidimensional non-uniformly hyperbolic maps with singularities commonly known as Viana maps. More precisely, these are obtained as C^3 small perturbations of the skew product ϕ_α of the cylinder $S^1 \times I$ given by*

$$\phi_\alpha(\theta, x) = (d\theta (\bmod 1)\,,\; 1 - ax^2 + \alpha \cos(2\pi\theta)),$$

where $d \geq 16$ is an integer, a is a Misiurewicz parameter for the quadratic family, and α is small. These maps admit a unique SRB measure μ (it is absolutely continuous with respect to $m = Leb$, has only positive exponents and $d\mu/dm \in L^p(m)$ where $p = d/(d-1)$) and are strongly topologically mixing on the attractor $\Lambda = \cap_{n\geq 0} f^n(S^1 \times I)$: for every open set A there exists a positive integer $n = n(A)$ such that $f^n(A) = \Lambda$. See [Via97, Alv00, AV02] *for more details.*

We show that (f, μ) satisfies the non-uniform specification using that the sequence of hyperbolic times is non-lacunar (that is $\frac{n_{k+1}-n_k}{n_k} \to 0$) at almost every x and that the image of hyperbolic balls grow to Λ after finitely many iterates.

First we observe that n_1 is integrable with respect to the SRB measure μ. In fact, since the tail of the first hyperbolic time map decays subexponentially fast with respect to m in particular one has $n_1 \in L^q(m)$ for every $q \geq 1$. Using once more Cauchy-Schwartz inequality and that $d\mu/dm \in L^p(m)$ it follows that $n_1 \in L^1(\mu)$ and, consequently, the sequence $n_j(\cdot)$ of hyperbolic times is non-lacunar (see e.g. [VV10, Corollary 3.8]*).*

The arguments used in the proof of Theorem C in [AV02] *give that for every image of a rectangle at a hyperbolic time grow to Λ after a finite number ℓ of iterates. Since Leb almost every point has infinitely many hyperbolic times then f is topologically exact. Therefore, the argument that (f, μ) satisfies the non-uniform specification property goes along the same lines used in proof of Lemma 6.1.*

We finish this section with some questions related with large deviations and weak specification properties. The first natural question is to investigate the connection between mixing and the non-uniform specification property.

Question 1: *Let (f, μ) be a non-uniformly expanding map with critical region and assume that f is topologically mixing. Does (f, μ) satisfy the non-uniform specification property? Same question, replacing the topologically mixing property by assuming that μ has exponential decay of correlations.*

It is also an interesting question to relate specification properties with the presence of discontinuities in the system. Indeed, Buzzi [Buz97] proved that contrary to

the characterization due to Blokh [Blo83] for continuous interval maps there exists a large class of topologically mixing but discontinuous maps of the interval (including β-transformations) so that the set of parameters for which the strong specification property holds although dense has zero Lebesgue measure. Since β-expansions have strong expansion, we believe that the simple presence of discontinuities should not be enough to prevent an affirmative answer to the following question.

Question 2: *Do all β-expansions of the interval satisfy the non-uniform specification property for all invariant measures? Is it true that "most" piecewise-expanding interval maps satisfy the non-uniform specification property?*

Note that an affirmative answer to the previous questions would be a contribution for a better understanding of the non-uniform specification property would give a wider class of examples for which our results apply. Finally, it remains the question of whether a large deviations principle can be established in a broad non-uniformly hyperbolic context with respect to not necessarily invariant weak Gibbs measures.

Question 3: *Given an invariant absolutely continuous expanding measure μ with exponential decay of the first hyperbolic time map does there exists a large deviation principle with respect to the (not necessarily invariant) weak Gibbs measure?*

In particular it would be interesting to give a precise description of the rate function in terms of the invariant expanding measures.

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